Mathematical knitting and crochet

Pat Ashforth and Steve Plummer
describe the way they use mathematical designs in knitting and crochet

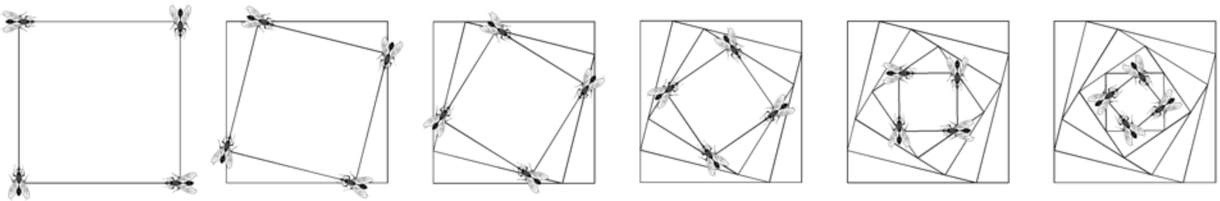

Figure 1,: Cyclic pursuit with four flies starting at the corners of a square

We have been designing and making knitted and crochet afghans (blankets) and hangings for many years, after being commissioned to create a geometric design for a US yarn company. When we realised that these large-scale items were ideal for teaching mathematics at all levels, we went on to produce instructions booklets so that other people could create their own. There are many mathematical techniques that can be used to create designs and the three described here are just a few of those you can find on our website (see *Where to find out more*). The popularity of these designs has encouraged many people to take an interest in geometry, even if they say they have a maths-phobia.

Our greatest achievement is an afghan called “Counting Pane” (pictured above), which was bought by the Science Museum. The first row represents the numbers 1 to 10, the second 11 to 20, and so on. The numbers from 1 to 10 each have a colour at the top of the column: 1 is blue, 2 is yellow, etc. If these numbers divide into the number of a square, the colour is shown in the square.

“Counting Pane” was inspired by the very familiar number squares found in many classrooms, where each multiplication table is shown on a separate sheet. We realised that they could be combined on something the size of an afghan so that the links between the various tables became more obvious. It also provokes discussion about how particular patterns arise, why some columns are more colourful than others, and how this can lead to the study of prime numbers.

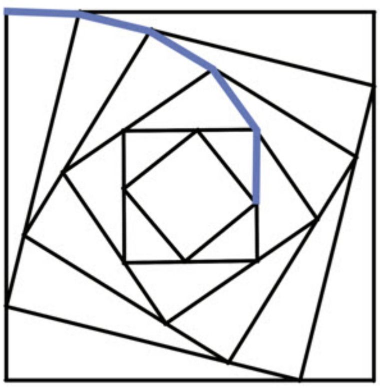

Figure 2: The triangles in the design, with a curve highlighted

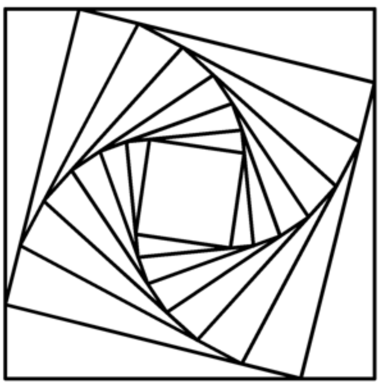

Figure 3: The triangles modified for knitting

Figure 4: Curve of pursuit afghan

Figure 5: Curve of pursuit cushion cover

Curves of pursuit

A pursuit curve is the path followed by the chaser in pursuit of the chased. This might be an animal pursuing its prey, or, maybe, one ship trying to catch another. Here we use a more sophisticated set of pursuits called cyclic or polygonal pursuits. This involves a number of creatures chasing one another in sequence with the last chasing the first, as, for example, with flies starting at the corners of a square, shown in figure 1. The path created by each fly in this case is a curve called an *equiangular spiral*. The curves become more obvious as more points along the route are connected, as shown in figure 2. If they were to keep going, they would eventually crash in the middle.

There is a certain amount of artistic (or mathematical) licence in the afghan design. On a real curve of pursuit, the flies travel at a constant speed. The construction diagrams in figures 1 and 2 measure equal lengths on the squares to indicate this constant speed and are an approximation of the curve since the flies are not actually following one another all the time. This design would be extremely difficult to replicate in knitting, as a separate set of instructions would be needed for each set of triangles. The knitted version uses the same shaping technique throughout, which results in triangles with equal angles, instead of a matching side. Thus the flies change their speed, but are always chasing one another, as shown in figure 3. The completed afghan and a smaller cushion version are shown in figures 4 and 5 respectively.

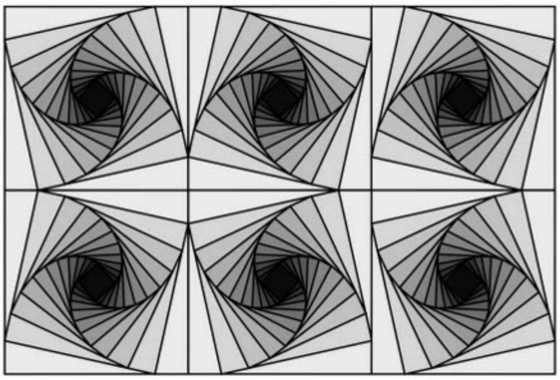

Figure 6: Designing with multiple curves of pursuit

In the summer of 2006, a photograph of an afghan made from our instruction booklet appeared in a book called Mason Dixon Knitting and fascinated knitters around the world became captivated by curves of pursuit.

Knitted curves of pursuit have been made in every colour combination imaginable. Three or four colours, or more, have been used and tonal variations also work well. There have even been some crochet versions. The design can be joined up, with or without reflection, in different ways, as shown in figure 6. This gives rise to op-art effects: you should be able to see “corn sheaves” and “fans” in this version. We have yet to see a version made of several cushion-sized pieces joined together. It is an idea we have looked at ourselves, though not actually made, but guess that someone somewhere will have tried it. Likewise with the hexagonal version shown in figure 7.

Dragon curves

We often use old maths books as a source of ideas. In the first instance, they are used as picture books, with little regard for the words. I have often been heard to say, “I could knit that,” in the most unlikely circumstances. One of our favourites is the *Penguin Dictionary of Curious and Interesting Geometry*. This was certainly the source of inspiration for our dragon curves. During a particularly boring exam invigilation, I was idly thumbing through the book, which I had done many times before, and the drawings of dragon curves leapt off the page. As soon as I got home that day, we, together with son Ben, started to play with the curves and they started to take over our lives.

The most time-consuming works we have made have been the dragon curve designs, as there are so many possible variations. Dragons are addictive. Everyone wants to have a go at adding them together, as the technique is so easy to master.

To make a paper dragon

Take a long strip of paper and fold it in half, in half again, then in half yet again. Open it up and crease it firmly on the folds, taking care not to bend them in the wrong direction. Stand the paper on its edge and you get the result shown in figure 8. If you turn over the strip of paper, the dragon will be going in the opposite direction. (At this stage it doesn’t bear much resemblance to a dragon, but as more folds are added it becomes obvious why they are so named.) All folding must be done in the same direction. When folding the paper, keep it on the table and always fold from the same end. The paper tends to curl slightly, but all the angles of the “mathematical dragons” are

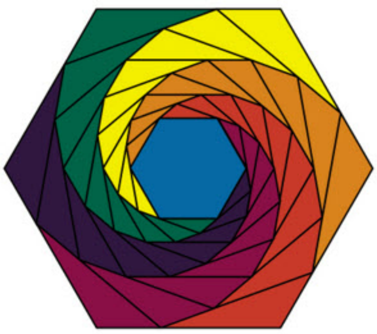

Figure 7,: Hexagonal curves of pursuit

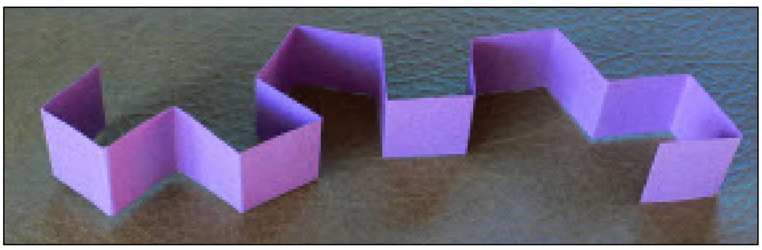

Figure 9: Folding a strip of paper to form a dragon

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technically right angles. (Dragons with other angles are possible but more difficult to represent in knitted and crochet objects.)

Each extra folding of a real strip of paper doubles the number of sections in the dragon. Dragons are described in terms of the number of folds you use to make them. This is called the *order* of the dragon: a single fold produces a dragon of order 1, two folds a dragon of order 2, and so on. Each section in the new dragon is only half the size of a section in the previous dragon. In each dragon there is one more section than corners. Figure 9 shows order 3, 4, and 5 dragons. The order 3 dragon has 8 sections and 7 corners, and the order 4 dragon has 16 sections. The paper dragon shown in figure 8 is an order 4 dragon with 16 sections. It lies so that the corners are distinct, whereas in the order 4 schematic dragon shown in figure 9 the right angles are preserved; they meet at some places, but they cannot cross. As the strip is folded more and more times, the folds pack closer and closer together, and the more folds there are, the more the plane is covered. For this reason, dragon curves are known as *space-filling curves*.

To find out what happens with a larger number of folds than is physically possible with paper, you need to draw them either on squared paper or by using one of the many computer drawing programs (see *Where to find out more*).

It is quite difficult to draw dragons at first: if you make a mistake, the rest of the dragon is wrong. However, if you understand how they fit together and can be taken apart, it becomes a great deal easier.

Figure 10 shows that two identical small dragons can be joined at right angles to each other and this process can be repeated until you get to the dragon you want. So two order 2 dragons give an order 3 dragon, and so on, up to order 6 (or further, if you need more). It often helps to think in terms of heads and tails. New dragons are created by joining tail-to-tail or head-to-head.

It is sometimes hard to see the path of the dragon, even though you know they can meet but they cannot cross. Figure 11 shows a order 9 dragon compared to a version where the corners have been rounded, so you can see which parts are joined and which are merely touching.

When dragons are created in this way (as they are for most of the crochet designs) each section of the "paper" remains the same length, whereas when the dragons are folded the lengths of the sections are halved with every fold.

Since they are made up of lines, once the dragons have been drawn and fitted together to cover the surface it is very easy to represent them in crochet. They are applied to a backing which can be any suitable square grid. The backing can be specially made using filet crochet, but, as most of it disappears from view when the dragons are applied, ready-made grids can be used, such as net curtains or rug canvas. You could even use a garden fence.

The dragons are applied through the holes in the backing and lie on the surface. Applying the first dragon is tricky. It takes a great deal of concentration to get all the twists and turns, but it soon becomes obvious when something is wrong, as the curves don't pack closely together, and it is very easy to undo and correct. When the first dragon is in place, the others curl around it. The cushion cover in figure 12 uses a mixture of different orders of dragon with the size changing. The rug designs in figure 13 show the effects that can be achieved by joining the dragons in different ways. In the top one they are joined head to head, in the middle tail to tail, and in the bottom head to tail.

Another aspect of dragons which has given rise to some knitting ideas is to look at the way the folds go on the opened out strip. They can be thought of as in-folds (`I`) and out-folds (`O`). The `I` and `O` could also stand for one and zero, if you find it easier to think in these terms. You could also think of lefts and rights, although that notation is not used here. The sequence of folds for the order 3 dragon (see figure 10) is `O O I O O I I`. Starting from the top, it goes in the same direction (out or left) for the first two folds, then it goes the other way once, then the first way twice, and, finally, the second way twice.

The next dragons described in this notation are:

order 4:
`OOIOOIIOOOIIOII`

order 5:
`OOIOOIIOOOIIOIIOOOIOOIIIOOIIOII`

The folds can also be represented by colours. Figure 14 shows the design for scarves of orders 3, 4, and 5, with the in-folds shown in blue and the out-folds shown in grey. Figure 15 shows the scarves.

It is difficult to spot the pattern at first, but it becomes more obvious if you identify the centre. The first action of folding makes a single crease in the

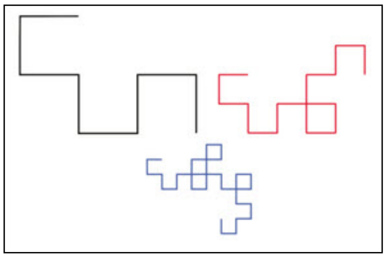

Figure 9: Folded dragons levels 3 to 5

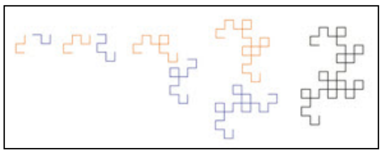

Figure 10: Constructing dragons, levels 3 to 6

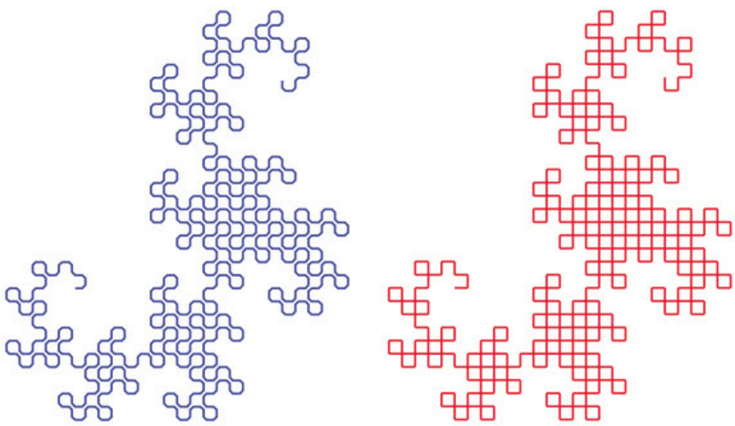

Figure 11: Two versions of a level 9 dragon

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centre of the paper. Every new folding adds the same number of extra folds on each side of the centre. There are always an odd number of folds. Figure 16 shows two ways to set this out to identify (a) folds which remain the same and (b) the sections in between folds created by successive foldings.

In the top part of figure 16, look at the sequence on each side of the centre `O`. Working out from the centre each `I` on the left becomes an `O` on the right and vice versa. It is easier to see how to create the next patterns if you push all the symbols over to the left-hand side as in the bottom half of figure 16. To make the order 5 sequence, write down the order 4 sequence, then an `O`, then the order 4 sequence backwards with each `I` changed to an `O` and each `O` changed to an `I`.

```
OOIOOIIOOOIIOII O
OOIOOIIIOOIIOII
```

When the sequence of folds for any dragon is known, the sequence for the next dragon can be generated by inserting terms before, between, and after those that are known. These additional terms are alternately O and I, with the first new term being the same as the first term of the previous dragon. For example:

```
Order 4  O O I O O I I O O O I I O I I
Order 5 OOIOOIIOOOIIOIIOOOIOOIIIOOIIOII
```

Similarly, it is possible to work backwards to find smaller dragons. For instance, if the sequence is known for order 5, strike out the first and every alternate term to give the sequence for order 4.

It is also possible to determine whether a fold is an `I` or an `O`. The following method will give exactly the same sequence as before, but the terms can be worked out one at a time or the nature of a chosen fold can be predicted independently. The method gives the sequence of folds created when the strip is folded from the right and the folds are counted from the left.

Step 1: Count the position of the fold from the left of the sequence
Step 2: Keep dividing the number by 2 until an odd number is reached
Step 3: Add 1
Step 4: Divide by 2

If the answer is even, it is an `I` fold; if the answer is odd, it is an `O` fold. So, for example, for the 20th fold of any dragon.

Step 1: Position 20
Step 2: 20 divided by 2 = 10; 10 divided by 2 = 5
Step 3: 5 + 1 = 6
Step 4: 6 divided by 2 = 3, which is odd, so the fold is an `O`

We have only covered dragons with right angle turns. It's not as easy to use other angles, but we leave the idea with you as a challenge.

Figure 12: A dragon cushion cover

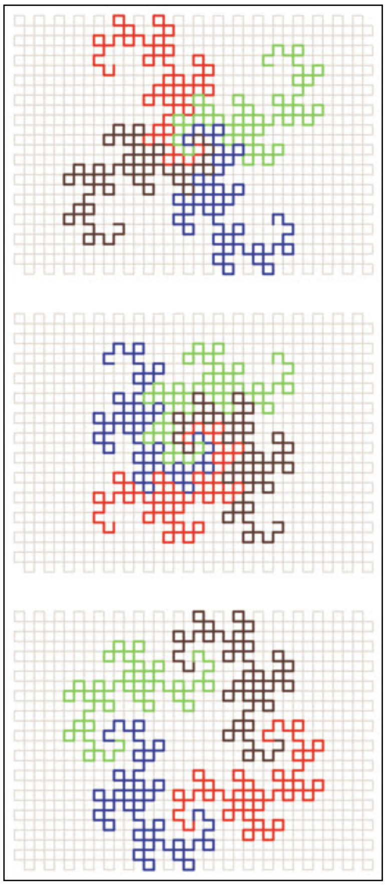

Figure 13: Dragon designs for rugs

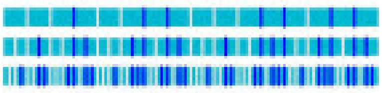

Figure 14: Designs for order 3, 4 and 5 scarves

Figure 15: Crochet order 3, 4 and 5 scarves

```
Order 1                O                1 fold
Order 2        O       O            I   3 folds
Order 3    O   O   I   O   O     I  I   7 folds
Order 4  O O I O O I I O O O I I O I I  15 folds
Order 5 OOIOOIIOOOIIOIIOOOIOOIIIOOIIOII 31 folds

Order 1 O
Order 2 OOI
Order 3 OOIOOII
Order 4 OOIOOIIOOOIIOII
Order 5 OOIOOIIOOOIIOIIOOOIOOIIIOOIIOII
```

Figure 16: Dragon curves, folding structure

Where to find out more

- Woolly Thoughts website is at woollythoughts.com, where you can also find our instruction books.
- There is also a Ravelry group for mathematical crochet and knitting at ravelry.com/groups/woolly-thoughts
- Paul J. Nahin, *Chases and Escapes: The Mathematics of Pursuit and Evasion.* Princeton University Press, 2007. This book deals with the mathematics of curves of pursuit.
- Kay Gardiner and Ann Shayne, *Mason Dixon Knitting: The Curious Knitters' Guide—Stories, Patterns, Advice, Opinions, Questions, Answers, Jokes and Pictures*. Potter Craft, 2006.
- Debbie New, *Unexpected Knitting*. Schoolhouse Press, 2003.
- For drawing on the computer, you could use the LOGO programming language. A sample Dragons program usually comes with most versions. Another program which includes Dragons and has many more such fractal curves is FRACTINT. Search the web to find free versions of these programs.